Optimal design of a cellular material encompassing negative stiffness elements for unique combinations of stiffness and elastic hysteresis

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HIGHLIGHTS
• We present a 3-spring model with a negative stiffness element for energy dissipation.
• An architected material implementation of the model is designed and fabricated.
• The performance of the architected material is modeled and verified experimentally.
• The geometry of the architected material is optimized for stiffness and damping.
• This tunable stiff damper can be easily manufactured in virtually any material.

GRAPHICAL ABSTRACT

ABSTRACT
Viscoelastic materials are commonly used to dissipate kinetic energy in case of impact and vibrations. Unfortunately, dissipating large amounts of energy in a monolithic material requires high combinations of two intrinsic properties – Young's modulus and loss factor, which are generally in conflict. This limitation can be overcome by designing cellular materials incorporating negative stiffness elements. Here we investigate a configuration comprising two positive stiffness elements and one negative stiffness element. This unit cell possesses an internal degree of freedom, which introduces hysteresis under a loading-unloading cycle, resulting in substantial energy dissipation, while maintaining stiffness. We demonstrate and optimize a simple implementation in a single material design that does not require external stabilization or pre-compression of buckled elements; these key features make it amenable to fabrication by virtually any additive manufacturing approach (from 3D printing to assembly and brazing) in a wide range of base materials (from polymers to metals). No additional intrinsic damping mechanism is required for the base material, which is assumed linear elastic. Furthermore, the architected material can be designed to be fully recoverable. When optimized, these architected materials exhibit extremely high combinations of Young's modulus and damping, far superior to those of each constituent phase.

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1. Introduction
The ability to dissipate kinetic energy resulting from impact or vibration is a key requirement for a number of applications in aerospace and mechanical engineering. This energy can be dissipated in a non-recoverable way, e.g., by plastic deformation [1,2], or in a recoverable manner, generally using viscoelastic materials [3]. Recoverable behavior is generally preferred, but the amount of energy that can be dissipated using viscoelastic materials is limited by the product of Young’s modulus and loss factor [4].

In recent years, a number of novel architected materials have been introduced that are based on elastic constituent phases, yet achieve hysteretic behavior under cyclic loading (and hence energy dissipation).
Notable examples are hollow micro and nano-lattices with extremely low density which dissipate energy through internal vibration following the local buckling of the hollow bars, while maintaining linear elastic response at the constituent level up to extremely large effective strains [5,6]. Unfortunately this mechanism requires extremely low relative densities, and hence is not applicable to materials with significant strength and stiffness requirements. An alternative design is provided by entangled-wire or woven materials, which damp vibration through internal friction [7,8]. Contrary to hollow microlattices, these architectured materials are generally very dense, resulting in heavier systems.

An energy dissipating mechanism that can overcome the challenges listed above is based on the snap-through buckling of elastic structural components with non-convex strain energy landscape [9,10], resulting in regimes of negative stiffness. Negative stiffness is a reversal of the usual directional relationship between force and displacement, which generally manifest itself in externally stabilized, displacement controlled structures over a range of displacements [11]: two pin-jointed springs with a non-zero initial angle, and an arch or a post-buckled beam undergoing snap-through (Fig. 1) are some examples of this concept.

The design of structures incorporating negative stiffness elements has been investigated over the past two decades, in particular with the goal of designing isolators [12–16]. Applications include seats [17, 18], earthquake protection [19] and ultra-sensitive optical devices [20]. The common configuration for a zero stiffness isolator consist of an arrangement of a positive and a negative stiffness element, combined in parallel. Tuning of the load-displacement curves for the springs can result in an effective load-displacement profile for the system featuring a flat plateau at the desired isolation force.

Recently, a number of investigators have studied simple arrangements of positive and negative stiffness elements to design structures and periodic architectured materials with extreme damping. The simplest configuration consists of a stack of non-convex strain energy elements arranged in series. Notable implementations of this concept are reported in [10, 19, 21,22] to design and realize cellular materials with high energy dissipation. In [23], superior damping performance was achieved by exploiting the unique post-buckling behavior of a free-end column. More complex mechanical models, consisting of series and parallel arrangements of positive and negative stiffness elements, have also been investigated [24–26]. These systems possess an internal degree of freedom that can be tuned to dramatically enhance the intrinsic loss factor of the base material under dynamic cyclic loadings. In [27], a clever implementation of a similar arrangement is used to realize an architectured material that simultaneously exhibits negative stiffness and negative Poisson’s ratio.

In this article, we propose and investigate a novel periodic cellular material configuration which allows independent control of stiffness and loss factor under cyclic loadings. In this novel design, each unit cell contains a particular arrangement of structural elements (henceforth called ‘3-spring configuration’), consisting in a positive stiffness element connected in series with a bi-stable (non-convex strain energy) element (possibly implemented by two pin-jointed springs at an angle), with the pair connected in parallel with a second positive stiffness element (Fig. 2a). This system of springs possesses an internal degree of freedom (point B), and consequently, when properly tuned, will exhibit hysteretic behavior under quasi-static cyclic loading, as schematically depicted in Fig. 2b. Clearly, the area under the hysteresis loop represents the energy dissipated in each cycle.

Although apparently similar to the models analyzed in [24–26], the proposed design is fundamentally different in that it dissipates energy via cyclic snap-through of the bi-stable element, as opposed to amplification of intrinsic damping. As discussed in detail in Section 2.5, this results in a loss coefficient that is nearly frequency-independent and largely independent on the intrinsic material damping.

We first develop a rigorous analytical model of the 3-springs configuration, obtaining the equations that characterize its behavior under cyclic quasi-static loading. Subsequently, we adapt the model to a specific cellular material implementation, and present a simple mechanical analysis that clearly reveals the effect of the geometric parameters of the unit cell on the effective mechanical response of the material (stiffness, damping, isolation stress). The model is validated against finite elements analyses and experimental characterization, performed on a 3D printed prototype. Finally, we incorporate the validated model in an optimization framework to identify the geometric parameters of Young’s modulus and damping, far superior to those of the constituent material. Whereas our analysis is purely quasi-static, we show that the conclusions apply to dynamic loadings as well, as long as the frequency of oscillation is lower than the natural frequency of the internal degree of freedom, which can be tuned over wide ranges by geometrical scaling and material selection. The dynamic behavior of this system

Fig. 1. Two examples of elements that can show negative stiffness, and their representative force-displacement behavior: (a) two pin joined springs arranged at an angle; (b) a sine shaped arch, with fixed end.
under frequencies that are comparable to the natural frequency of the internal degree of freedom is extremely complex and outside the scope of this work: it will be considered in future studies.

2. Mechanical response of negative stiffness-based cellular materials

2.1. Mechanical model of a 3-spring system

Consider the mechanical model in Fig. 2a. It consists of a linear spring of positive axial stiffness \( K_1 \) arranged in series with a non-linear element consisting of two linear springs of axial stiffness \( K_2 \), pin jointed and oriented at an angle \( \theta \) from the horizontal plane. Under vertical loads, the two springs of stiffness \( K_2 \), arranged as described, form a non-linear element with the force-displacement behavior including a region of negative stiffness (Fig. 1a). The \( K_1-K_2 \) system is arranged in parallel with another linear spring of axial stiffness \( K_3 \). The springs \( K_1 \) and \( K_3 \) are assumed rigidly connected on their top nodes, with only one axial degree of freedom, and the bottom nodes of the springs \( K_2 \) are assumed pin jointed. The entire system, of width \( 2W \), is loaded with a cyclic vertical force \( F \) (assumed positive if pointing downwards). The displacement of point A is denoted \( x_1 \), whereas the displacement of point B is denoted \( x_2 \). We refer to this mechanical system as the ‘3-spring configuration’.

The qualitative behavior of the system under cyclic vertical loadings (Fig. 2b) is easily understood. When the system is compressed from its rest stage, the spring \( K_1 \), if sufficiently compliant, will start accumulating strain until it can exert enough force to induce snap-through in the negative stiffness component, i.e. the assembly of 2 springs of stiffness \( K_2 \); after snap-through, the spring \( K_1 \) will release nearly all the strain; subsequently, when the load direction is reversed, a similar effect will occur, i.e. tensile stress build-up in the spring \( K_1 \) until snap-through of the system of springs \( K_2 \). Hence, the key to understand this hysteretic behavior is a detailed analysis of the mechanical response of the combination \( K_1-K_2 \).

The effective static stiffness of the 3-spring configuration, \( K_e \), is defined as the initial slope of the \( (F, X_1) \) curve; the effective loss factor, \( \psi_r \), is defined as the ratio of the dissipated energy \( (E_{\text{diss}}) \) to the energy input \( (E_{\text{inp}}) \), i.e., \( \psi = E_{\text{diss}}/E_{\text{inp}} \).

To quantitatively analyze the behavior of the 3-spring configuration, we initially focus on the equilibrium between \( K_1 \) and \( K_2 \), schematically represented in Fig. 3a. As before, we define \( X_1 \) the displacement of the top point (A) and \( X_2 \) as the displacement of the internal degree of freedom (B); \( F \) is the external force applied to the entire \( K_1-K_2 \) system. If \( f_{k_1} \) is the force in the spring \( K_1 \), and \( f_{k_2} \) the vertical force in the system \( K_2 \), equilibrium at point B implies: \( f_{k_2} = f_{k_1} \approx 0 \). This condition can be used to identify the force \( F \) and the displacement \( X_1 \) at which

\[
f_{k_2} = f_{k_1} = K_1 \cdot (X_1 - X_2) = 0\]

and

\[
f_{k_1} = f_{k_2} \approx 0 = K_1 \cdot X_1 = K_2 \cdot X_2 \approx 0.
\]

For the system shown in Fig. 3b, with \( f_{k_1} \) and \( f_{k_2} \) represented as a straight line of slope \( -K_1 \), with intercept increasing upon loading. The equilibrium point is the intersection of \( f_{k_1} \) and \( f_{k_2} \). This equilibrium is possible up to the tangency condition depicted in Fig. 3b (point i), after which sudden snap-through to point ii occurs.

Upon unloading, the system remains stable up to point iii, after which it snaps to point iv. Hence, if we denote displacement and force at points i and iii as \((X_1^i, f^i)\) and \((X_2^i, f^i)\), respectively, these critical values can be obtained as:

\[
dF/dX_2^{i} = -K_1 f^i = f_{k_2} (X_2 = X_1^i) X_2^i \in [0, W \tan \theta_0]
\]

and

\[
dF/dX_1^{i} = -K_1 f^i = f_{k_2} (X_2 = X_1^i) X_2^i \in [W \tan \theta_0, 2W \tan \theta_0]
\]

As \( f_{k_2} \) is antisymmetric with respect to \( X_2 = W \tan \theta_0 \), clearly we have: \( f^i = -f^i \), and \( X_2^i = 2W \tan \theta_0 - X_2^i \). We can plot the behavior of the entire \( K_1-K_2 \) system in the \( F-X_1 \) domain (Fig. 3c): \( X_1^\text{max} \) and \( X_1^\text{min} \) can be easily found as:

\[
K_1 \cdot (X_1^\text{max} - X_2^i) = f^i \Rightarrow X_1^\text{max} = X_2^i + f^i/K_1
\]

and

\[
K_1 \cdot (X_1^\text{min} - X_2^i) = f^i \Rightarrow X_1^\text{min} = X_2^i + f^i/K_1
\]
Notice that the loading curve in Fig. 3c is not exactly a straight line, due to the shape of the load-displacement curve of $K_2$. However, it is generally very close to a straight line. The addition of the spring $K_3$ to the 3-spring system has a simple stiffening effect, i.e.:

$$f_{\text{max}} = f^+ + 2 \cdot K_3 \cdot X_1^{\text{max}}$$ \hspace{1cm} (5)

$$f_{\text{min}} = f^- + 2 \cdot K_3 \cdot X_1^{\text{min}}$$ \hspace{1cm} (6)

resulting in the overall mechanical response of the entire 3-spring system depicted in Fig. 2b.

From Eq. (2) and Fig. 3b, we can extract a necessary condition for hysteretic behavior, i.e.: $K_1 \cdot |dF/dX_2| \sim \omega \cdot \tan \theta$; this imposes that the spring $K_1$ must be more compliant than the maximum stiffness of the system $K_3$ in its negative stiffness region.

For most cyclic load applications, we want to request that the load-displacement behavior be recoverable, i.e., that the force $F_{\text{min}}$ be positive. This condition is easily expressed as $F_{\text{min}} = f^- + 2 \cdot K_3 \cdot X_1^{\text{max}} > 0$.

Finally, we can express the dissipated energy as:

$$E_{\text{diss}} = \left( f^+ - f^- \right) \left( X_1^{\text{max}} - X_1^{\text{min}} \right) - \left( X_1^{\text{max}} - X_1^{\text{min}} \right) \frac{f_{\text{max}}}{X_1^{\text{max}}}$$ \hspace{1cm} (7)

The loss factor can then be expressed as:

$$\psi = \frac{E_{\text{diss}}}{\frac{1}{2} f_{\text{max}} \cdot X_1^{\text{max}}}$$ \hspace{1cm} (8)

If we assume an architected material with unit cell schematically depicted in Fig. 2a, the effective Young’s modulus of the material can be written as:

$$E = \frac{f_{\text{max}}}{X_1^{\text{max}}/H} = \frac{X_1^{\text{max}}}{f_{\text{max}}/2WD} \cdot \frac{H}{X_1^{\text{max}}} \cdot \frac{f_{\text{max}}}{2WD}$$ \hspace{1cm} (9)

where $W$, $H$ and $D$ represent the semi-width, height and depth of the unit cell.

The product of the Young’s modulus and the loss factor is a key figure of merit for stiff dampers, i.e.:

$$\Pi = E \cdot \psi = \frac{f_{\text{max}}}{X_1^{\text{max}}} \cdot \frac{H}{2WD} \cdot \frac{E_{\text{diss}}}{\frac{1}{2} f_{\text{max}} \cdot X_1^{\text{max}}} = \frac{2 \cdot E_{\text{diss}}}{(X_1^{\text{max}})^2} \cdot \frac{H}{2WD}$$ \hspace{1cm} (10)

This analytical model can be graphically expressed in the form of design maps, visualizing the influence of all geometric parameters on the energy dissipation. The geometric variables are non-dimensionalized a $X_1^{\text{max}}/K_1$, $X_1^{\text{max}}/K_3$, $X_1^{\text{max}}/W$, $X_1^{\text{max}}/W$ and $\theta_\circ$.

Fig. 4 presents contour plots of the normalized figure of merit, $\Pi = E \cdot \psi \cdot 2WD/\Pi K_1$, as a function of $K_3/K_1$ and $K_3/K_1$ for four different values of the initial angle $\theta_\circ$. The dashed areas in the plots correspond to conditions in which $E_{\text{diss}}$ is negative, resulting in non-recoverable behavior.

From these maps, the following important observations can be extracted:

(i) An increase in the geometric parameter $K_3/K_1$ always results in increased values of the figure of merit, $\Pi$.

(ii) The geometric parameter $K_3/K_1$ does not significantly affect the magnitude of the figure of merit; this is because while the effective Young’s modulus of the system is directly proportional to $K_3/K_1$, the loss factor is inversely proportional to the same geometric parameter (this conclusion can be derived from Equations (10) and (7)), and the two effects approximately cancel out.

(iii) As the initial angle $\theta_\circ$ is increased, the vertical force exerted by the spring system $K_3$ increases, and hence the parameter $K_3/K_1$ has to be reduced in order to achieve the same value of the figure of merit, $\Pi$. An important implication is that the maximum value of $K_3/K_1$ that ensures a recoverable system decreases as $\theta_\circ$ is increased.

(iv) Recoverability is strongly affected by the parameter $K_3/K_1$; the smaller this parameter, the smaller the recoverability range.

(v) The maximum values of the figure of merit $\Pi$ are achieved for values of the parameter $K_3/K_1$ that are at the cusp of non-recoverability. In other words, near-optimal recoverable systems are very difficult to design, as any small geometric defects might result in loss of recoverability.
of the semicircular arches implementing the springs or complex assembly, all springs are implemented as arches. The shapes fabrication in a single material system without the need for joints and/together with a 3D printed unit cell are shown in Fig. 5. To enable simple implementation of the 3-spring model described in the previous section.

We conclude this section by emphasizing that these important observations are general and apply to any physical implementation of a cellular material (or a structure) that is mechanically represented by the 3-spring configuration depicted in Fig. 2a. All these conclusions are very useful in informing the choice of a practical implementation with near-optimal performance.

2.2. Mechanical model of a cellular material implementation

In this section, we propose and characterize a cellular material implementation of the 3-spring model described in the previous section. A schematic drawing of the unit cell and the multi-cell material, together with a 3D printed unit cell are shown in Fig. 5. To enable simple fabrication in a single material system without the need for joints and/or complex assembly, all springs are implemented as arches. The shapes of the semicircular arches implementing the springs $K_1$ and $K_3$ are chosen to allow large strains in the linear regime without yielding or buckling. Clearly, this poses a limit on the stiffness of the entire system, as both springs are bending-dominated. The implications of this decision are detailed in the following sections. The $K_2$ two-spring assembly is implemented as a double arch, as opposed to the more common pre-buckled beam; differently from a pre-buckled beam, the arch does not require external pre-compression, thus dramatically simplifying the fabrication of a cellular material design; the choice of a double arch as opposed to a single arch ensures symmetric snap-through, without rotation of the central point, hence avoiding less snappy anti-symmetric modes.

The behavior of an arch under compression is qualitatively similar, but not quantitatively identical, to the two-spring $K_2$ model analyzed in Section 2.1. A mechanical model of the $K_1 - K_2$ system with the arch implementation is depicted in Fig. 6. The behavior of an arch (deforming via mode 3 snap-through instability) was derived in [28]. The key difference relative to the 2-spring system is that the force-displacement curve for the arch has a sharp transition between positive and negative stiffness regions. Hence we simply have: $X_f^2 = d_{top}$, $X_f^3 = d_{bot}$, $f^+ = 2 \cdot f_{top}$ and $f^- = 2 \cdot f_{bot}$ (see Fig. 6b), where the factor 2 in the force terms results from the choice of a 2-arch spring. Equilibrium of the $K_1 - K_2$ system imposes:

\[
2 \cdot K_1 \cdot (X_{1,\text{top}} - d_{top}) = 2 \cdot f_{top} \tag{11}
\]

\[
2 \cdot K_1 \cdot (X_{1,\text{bot}} - d_{bot}) = 2 \cdot f_{bot} \tag{12}
\]

The factor of 2 reflects the presence of two springs. From Eqs. (11) and (12) we readily obtain:

\[
X_{1,\text{top}}^{\text{max}} = d_{top} + \frac{2 \cdot f_{top}}{2 \cdot K_1} \tag{13}
\]

\[
X_{1,\text{bot}}^{\text{min}} = d_{bot} + \frac{2 \cdot f_{bot}}{2 \cdot K_1} \tag{14}
\]

The condition that ensures hysteretic behavior in this case can be expressed as: $-K_1 \geq (f_{bot} - f_{top})/(d_{bot} - d_{top})$; in other words, $K_1$ has to be more compliant than the absolute value of the negative stiffness of the double arch.

If we now introduce the additional positive stiffness springs $K_3$, the maximum and minimum forces in the system are given by:

\[
F_{\text{max}} = 2 \cdot f_{top} + 4 \cdot K_3 \cdot X_{1,\text{top}}^{\text{max}} \tag{15}
\]

\[
F_{\text{min}} = 2 \cdot f_{bot} + 4 \cdot K_3 \cdot X_{1,\text{bot}}^{\text{min}} \tag{16}
\]
The dissipated energy can be expressed as:

\[ E_{\text{diss}} = \left( F_{\text{max}} - F_{\text{min}} \right) \left( X_{1_{\text{max}}} - X_{1_{\text{min}}} \right) - \left( X_{1_{\text{max}}} - X_{1_{\text{min}}} \right)^2 \frac{F_{\text{max}}}{X_{1_{\text{max}}}} \]

\[ = \left( 2 \cdot f_{\text{top}} - f_{\text{bot}} \right) \left( X_{1_{\text{max}}} - X_{1_{\text{min}}} \right) - \left( X_{1_{\text{max}}} - X_{1_{\text{min}}} \right)^2 \frac{F_{\text{max}}}{X_{1_{\text{max}}}} \]  

(17)

with the loss factor following as: \( \psi = E_{\text{diss}}/(F_{\text{max}} \cdot X_{1_{\text{max}}}/2) \).

The expressions for the effective Young’s modulus is given by:

\[ E = \frac{F_{\text{max}}}{L^2} = \frac{F_{\text{max}}}{X_{1_{\text{max}}} \cdot L} = \frac{2 \cdot f_{\text{top}} + 4 \cdot K_3 \cdot X_{1_{\text{max}}}}{X_{1_{\text{max}}} \cdot L} \]

(18)

with \( L \) the size of the cubic unit cell.

Once again, we define the key figure of merit for our architected material as the product of the Young’s modulus and the loss factor of its unit cell, i.e., \( \Pi = E \cdot \psi \).

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**Fig. 5.** Architected material implementation of the 3-spring configuration depicted in Fig. 2. (a) Geometry of each spring element; (b) CAD model of a unit cell; (c) CAD model of a three-dimensional lattice; (d) physical implementation of a unit cell, produced by Fused Deposition Modelling.

**Fig. 6.** Analysis of the \( K_1 - K_2 \) equilibrium for an arch implementation of \( K_2 \). (a) Sketch of the elements and definition of variables; (b) force-displacement diagrams for the equilibrium of point B: \( f_{\text{top}} \) is in blue, \( f_{\text{arch}} \) is in red; (c) force-displacement curve for the entire 2-spring system (equilibrium of point A).
We now need to express the spring stiffness $K_1$, $K_2$ and $K_3$ as a function of the geometric parameters of its implementations. Details are provided in Appendix A. With reference to the geometric variables defined in Fig. 5a, the results are:

\[
K_1 = aEw_1 \left( \frac{t_1}{t_1} \right)^3
\]
\[
K_3 = bEw_3 \left( \frac{t_3}{t_3} \right)^3
\]
with $E$ the Young’s modulus of the parent material, $a = 1.4$ and $b = 0.46$.

The mechanical response of the $K_2$ arch is fully characterized by two force values ($f_{\text{top}}$ and $f_{\text{bot}}$) and two displacement values ($d_{\text{top}}$ and $d_{\text{bot}}$). These are given by [28]:

\[
f_{\text{top}} = \left( 8 \pi^4 - 6 \pi^4 \frac{d_{\text{top}}}{h} \right) \frac{E_h t_1^2 W_2}{12 t_1^3}
\]
\[
f_{\text{bot}} = \left( 8 \pi^4 - 6 \pi^4 \frac{d_{\text{bot}}}{h} \right) \frac{E_h t_1^2 W_2}{12 t_1^3}
\]
\[
d_{\text{top}} = h \left( 1 - \sqrt{\frac{16}{3(h/t_1)^2}} \right)
\]
\[
d_{\text{bot}} = h \left( 1 + \sqrt{\frac{16}{3(h/t_1)^2}} \right)
\]

For proper operation, we must clearly impose that the material remain elastic over the entire cycle, up to the limit force $f_{\text{top}}$. Imposing this condition requires an expression for the maximum stress in each spring element as a function of the displacement $X$. Refer to Appendix A for details. For spring elements $K_1$ and $K_2$ the maximum stresses are given by:

\[
\sigma_{1\text{max}} = c E \frac{t_1}{t_1} (X_1 - X_2) + d E \frac{t_2}{t_1} (X_1 - X_2)
\]
\[
\sigma_{2\text{max}} = e E \frac{t_1}{t_1} X_1
\]
with $c = 2.1$, $d = 1.4$ and $e = 3/m$.

The maximum stress in the arch $K_2$ is reached during snap-through and can be expressed as [28]:

\[
\sigma_{2\text{max}} = \frac{\pi^2 E \frac{t_2 h}{L} \left( 2 + \frac{4 t_2}{3 h} \right)}{L}
\]

The conditions that all stresses remain below the yield strength of the material will be imposed as constraints in the optimization algorithm, to ensure that all springs remain elastic over the entire range of motion.

2.3. Numerical modeling and verification

In order to verify the simple analytical model discussed in the previous section, a Finite Elements simulation is performed with the commercial software Abaqus. A unit cell conceptually identical to that in Fig. 5b is modeled with shell elements (S4R). To simplify meshing, the $K_3$ spring is modeled as two loops, as opposed to the four loops shown in Fig. 5b; similarly, the $K_2$ double arch depicted in Fig. 5b is modeled as a single arch and rotated. The widths and thicknesses of all the elements are properly adjusted to result in the desired stiffness. The top and bottom frames of the unit cell are not explicitly modeled; rather, all springs are fully clamped at the top and bottom sides. The top side is uniformly displaced in the negative vertical direction, resulting in uniaxial compression. Capturing the large instabilities that occur upon snap-through is a considerable challenge for traditional quasi-static Finite Elements algorithms. Although dynamic explicit algorithms could in principle be used, their conditional stability requires extremely time-intensive simulations to model deformation rates that are slow enough to approximate the quasi-static response. To address these challenges, we use the Riks algorithm, a quasi-static numerical scheme that was specifically designed to model post-buckling response. In this algorithm, both the load and displacement increments at each iteration are treated as variables (unlike in load or displacement controlled static simulations). Therefore, another quantity must be used to measure the progress of the solution: Abaqus uses the “arc length”, which is defined in order to follow the tangent stiffness of the system extracted from the previous iteration. Upon snap-through, the load suddenly drops and the algorithm automatically inverts the direction of the displacement increment in order to maintain convergence. During this process, energy is artificially dissipated. When the system reaches stability again, the displacement increment returns to the original direction and the loading cycle continues. A similar phenomenon occurs upon unloading. In order to ensure convergence, we limit the arc length in the range $[10^{-100} \text{ mm}–0.08 \text{ mm}]$, with the maximum number of iterations set to $10^7$.

The geometry of the unit cell is chosen as follows:

\[
K_1 \begin{cases}
   t_1 = 2.1 \text{ mm} \\
   t_2 = 1.1 \text{ mm} \\
   W_1 = 9.7 \text{ mm} \\
   W_2 = 30 \text{ mm} \\
   L_1 = 100 \text{ mm} \\
   L_2 = 10 \text{ mm}
\end{cases}
\]

\[
K_2 \begin{cases}
   t_3 = 2.6 \text{ mm} \\
   R_3 = 39.8 \text{ mm} \\
   W_3 = 10.2 \text{ mm}
\end{cases}
\]

The material is modeled as linear elastic, with Young's modulus, $E = 174040 \text{ MPa}$ and mass density, $\rho = 1100 \text{ kg/m}^3$. These values are representative of acrylonitrile butadiene styrene (ABS) printed in a Flashforge Creator Pro Fused Deposition Modeling machine, with fiber orientation along the loading direction (see Appendix B). As the unit cell is designed to undergo snap-through without yielding, no plastic properties are defined.

The load-displacement curve for the unit cell obtained by the Finite Elements simulation is depicted in Fig. 7, alongside the analytical prediction. Images of the deformed mesh with Von Mises stress contours at three points in the simulation ((1) along the loading path, (2) immediately before snap-through, and (3) after snap-through) are shown in the insets. Notice that the numerical load-displacement curve has a S-shape. This is characteristic of the Riks post-buckling analysis; although those are equilibrium positions, there are unattainable in a load or displacement controlled experiment. A displacement controlled loading-unloading experiment would result in a sudden load decrease at snap-through upon loading, and a sudden load increase at snap-through upon unloading (dashed vertical lines).

Note that the FE simulation agrees quite well with the analytical prediction in terms of stiffness, shape/size of the hysteresis cycle, and loads at snap-through events. The displacement width of the hysteresis cycle is slightly overpredicted by the analytical model. This discrepancy is primarily attributed to an overestimation of the rigidity of the torsional boundary conditions for the arch. Overall, though, this simulation verifies the accuracy of the analytical model.

2.4. Fabrication and experimental validation

A unit cell sample with the topology represented in Fig. 5b, and geometry notionally identical to that employed in the FE analyses, is additively manufactured in the polymer acrylonitrile butadiene styrene (ABS) via Fused Deposition Modeling (FDM). A Flashforge Creator Pro
is used for all prints. The three spring elements are printed separately, to ensure that the filaments of ABS are aligned with the direction of maximum loading for each spring. The fully assembled unit cell is shown in Fig. 5d (different colors are used for visual clarity, although the base material is the same throughout).

The mechanical response of the base material is characterized by uniaxial loading of a dog bone sample of ABS printed with filaments along the loading direction: elastic modulus and yield strength of the base material are extracted from the uniaxial stress-strain curve (see Appendix B for details).

The fully assembled unit cell is tested in uniaxial compression under displacement control. A displacement of 15 mm is applied at constant rate of 0.1 mm/s; this rate is chosen to be faster than the rate used in the tensile test of the parent material (Appendix B), in order to minimize viscoelastic effects. Soon after the occurrence of snap-through, the displacement direction is reversed and the compression platen is brought back to the original position.

The load-displacement curve for the whole cycle is depicted in Fig. 7, alongside the analytical and numerical predictions. The following observations can be made: (a) The stiffness of the sample is perfectly captured by the numerical model. Both experimental and numerical curves show a decrease in slope upon loading (which is not predicted by the analytical model); in principle, this could be due to viscoelastic behavior of the polymer or geometric non-linearities. However, as the numerical model assumes linear elastic behavior with constant Young’s modulus and no time dependence, we conclude that this feature is associated with geometric non-linearities, which are well captured by the numerical model. (b) The magnitude of the load drop upon snap-through (and subsequent load increase upon reverse snap-through) are well captured by the models. (c) The area of the hysteresis cycle (energy dissipation) is under-predicted by the models, by ~16%. This is attributed to the viscoelastic behavior of the polymer, whereby some amount of energy is dissipated in the polymeric material itself. (d) The displacements at which snap-through occurs upon loading and unloading are under-predicted by the models, by ~17%. This is attributed to imperfect fit among the various components and general manufacturing imperfections.

Overall, the experiment on a single unit cell validates the analytical model to a degree that is more than adequate for optimal design studies. In the next section, we will investigate the optimal design of an architected material based on this architecture with exceptional combinations of stiffness and energy dissipation.

2.5. Upper bound on excitation frequency

In traditional damping materials (e.g., viscoelastic solids), the energy loss characteristics are frequency dependent, due to the viscous nature of the loss mechanism. One very important aspect of the bi-stable system investigated in this work is that energy loss is to first order not frequency dependent, but rather amplitude dependent. The amplitude sensitivity arises due to the bi-stability of the system and the existence of a critical load or displacement that triggers snap-through. For the envisioned lattice material, the sensitivity to stroke and amplitude strongly depend on the choice of the design parameters, i.e. different combinations of elastic constants, as well as interactions between neighboring elements. Such interaction effects are out of scope for this paper, although they could be studied with the mechanical analysis presented herein.

The frequency-independence of the loss coefficient in the bi-stable system analyzed in this work is subject to a very important caveat: the forcing frequency must be much lower than the natural frequency of oscillation of the internal degree of freedom. As long as this condition is met, the energy stored in the elastic system immediately prior to snap-through is quickly dissipated by high frequency vibrations of the internal degree of freedom, which strongly enhance any loss mechanisms present in the system (e.g., viscoelastic damping within the material, viscous damping by the air, structural damping at the connectors, etc...). This results in the ‘clean’ hysteresis cycles seen in Fig. 7. If the forcing frequency of oscillation approaches the natural frequency of the internal degree of freedom (henceforth denoted as $f_r$), a much more complex regime ensues, for which the quasi-static analysis presented in this work is no longer adequate (although substantial energy dissipation is still possible). Although certainly interesting both scientifically and technologically, a detailed analysis of this complex regime is far outside of the scope of this work. The implication is that the quasi-static results presented in this work are (i) essentially frequency independent and (ii) valid up to a critical frequency, $f_c$. We now proceed to estimate this critical frequency.

The dynamics of the internal degree of freedom are determined by the elastic constants and mass characteristics of the elements. Before
and after a snap-through event, the internal degree of freedom has a
dynamic response that can be represented by a lumped model that
includes the effective stiffness of the upper spring (\(K_1\)), the effective
stiffness of the arch (\(K_2\)) and a lumped effective mass in the center.
This element then has a resonant frequency given by:

\[
f_r = \frac{1}{2\pi} \sqrt{\frac{K_1 + K_2}{m_{eff}}} \quad \text{and} \quad m_{eff} = \alpha_1 m_1 + \alpha_2 m_2 \quad \text{and} \quad 0 < \alpha < 1
\]

(26)

where \(K_1\) is given by Eq. (19), \(K_2\) is given by Eq. (21), \(m_1\) and \(m_2\) are the
masses of the two springs, and \(\alpha_1, \alpha_2\) are non-dimensional geometric
parameters of order 1. If the states are similar before and after snap
through, then the impulse in both directions will create an oscillation
parameters of order 1. If the states are similar before and after snap
through, then the impulse in both directions will create an oscillation
oscillation below.

damped down by the intrinsic and extrinsic loss mechanisms described
through, then the impulse in both directions will create an oscillation
parameters of order 1. If the states are similar before and after snap
through, then the impulse in both directions will create an oscillation

A realistic lower bound for the resonant frequency can be obtained
by assuming \(\alpha_1 = \alpha_2 = 1\) in Eq. (26). With this assumption and the geo-
dmetric dimensions and materials properties reported above, the reso-
nant frequency of the internal degree of freedom for the prototype
depicted in Fig. 5d can be estimated as 407 Hz (see Eqs. (19) and
(21)). This frequency is over 5 orders of magnitude higher than the exci-
ting frequency (~0.003 Hz), which explains why the hysteresis loops
in Fig. 7 show almost perfectly sharp snap-through events. A finite ele-
ments analysis of the \(K_1 - K_2\) system was also conducted to verify the
assumptions made in estimating the effective mass in Eq. (26). The FE
results provide a resonant frequency of 500 Hz, consistent with values of
\(\alpha_1 = \alpha_2 = 0.66\). Hence the simplified analytical model is overly con-
servative by ~20%.

Importantly, Eq. (26) can be used to estimate the effect of geometric
scaling and materials selection on the natural frequency of the internal
degree of freedom, and hence the maximum operating frequency of
the arch system. If we scale all dimensions by a factor \(\beta\) (i.e., \(r_1 - 1 = \beta r_1^1, W_1 = \beta W_1^1, W_2 = \beta W_2^2, W_3 = \beta W_3^3, \ldots\)),
where \(r_1^1, r_2^2, W_1^1, W_2^2, W_3^3, \ldots\) are the original dimensions,
then simple scaling arguments show that all masses scale as \(\beta^3\), whereas all stiffness scale as \(\beta\). The implication from Eq. (26) is that the resonant frequency of the internal degree of
freedom scales as \(1/\beta\). Similarly, if we replace our material with density
\(\rho_0\) and modulus \(E_0\) with a different material with density \(\rho\) and modulus
\(E\), the resonant frequency of the internal degree of freedom changes by
\(\sqrt{E/\rho}/\sqrt{E_0/\rho_0}\). As an example, if we fabricate our device in aluminum
with a unit cell size of 1 cm, the maximum operating frequency rises to
~11 kHz. For many applications, this value is well above the expected
operating frequencies.

3. Optimal design

3.1. Statement of the optimization problem

We seek to identify the optimal dimensions for the architected ma-
terial schematically represented in Fig. 5 for maximum combinations of
stiffness and loss factor. Specifically, the objective function will be the
figure of merit, \(I = E \cdot \psi\) introduced in Section 2. We assume that
the unit cell has external dimensions \(L_x \times L_x \times L_z\) and the design variables
are the dimensions \(t_1, t_1, W_1, W_2, h, W_3, t_3, R_3, W_4\) (see Fig. 5a for variable
definition). Two constraints need to be imposed: (i) none of the struc-
tural elements are allowed to yield during the entire loading-unloading
cycle, and (ii) the unit cell is required to be recoverable, i.e., it must
return to its initial configuration upon unloading. Solving this problem
numerically as stated presents numerous computation challenges. In
particular, infinitely many combinations of the geometric variables
exist that result in the optimal configuration. As an example, consider
the spring \(K_1\), which is defined by 3 geometric variables \((t_1, t_1, W_1)\):
then the optimal value of the stiffness \(K_1\) is determined, and the
condition of no yielding, i.e., \(\sigma_{\text{yield}}^{max}(t_1, t_1) < \sigma_0\) is imposed.
infinity many combinations of \((t_1, t_1, W_1)\) exist that satisfy all conditions. The
same is true for other springs. Besides, each value of the objective
function \(I\) can be obtained by infinite combinations of \(E\) and \(\psi\). All
this results in an infinite set of optimal solutions, which make this
optimization problem challenging to solve numerically. Furthermore,
the no yielding conditions are non-linear (see Eqs. (22)–(24)), which
results in significant computational cost.

To make the numerical optimization more tractable, we make two
important modifications to the problem statement:

1) We treat the stiffness of the spring \(K_1\) as an independent variable
(defined within a reasonable range), so that the values of its geo-
metric parameters \((t_1, t_1, W_1)\) do not enter the optimization prob-
lem. Once the optimal value of \(K_1\) is identified by the optimizer,
we apply the non-linear no-yielding condition to the element \(K_1\)
to identify a suitable set of geometric parameters. Clearly, the
advantages of this approach are two-fold: first, we reduce the
number of variables from 6 to 4 and eliminate a nonlinear con-
straint, thus reducing the computational cost of the optimization
algorithm; second, we eliminate the infinite set of local optima
related to the under-determination of the geometry of \(K_1\).

2) Rather than using the figure of merit, \(I = E \cdot \psi\), directly as the
objective function, we sweep the Young’s modulus of the cell, \(E\),
within a prescribed range, and for every value of \(E\) we solve for the
geometry that maximizes \(\psi\). As \(K_1\) (and hence it geometric variables
\(t_1, R_3, W_4\)) only affect \(E\) (and not \(\psi\)), this approach further
reduces the number of variables.

Hence, the statement of the optimization problem is the following:

For any given value of \(E\) within a prescribed range, we find the values of the variables \(K_1, t_2, h, W_3\) that maximize \(\psi\), subject to the follow-
ing constraints: (a) no yielding of the arch (spring \(K_2\)); (b) recover-
ability under a compression cycle, i.e., \(\sigma_{\text{yield}}^{max}<0\); (c) \(h/t_2 > 2.31\) (a
condition reported in [28] to ensure vertical symmetry in the buck-
ling mode of the arch, important for energy dissipation). The yield
strain and Young’s modulus of the parent material, \(E_0\) and
\(\psi_0\), and the overall size of the unit cell, \(L_x\), are assumed fixed parameters.

As we want to ensure that the optimal geometry could be 3D printed
with a FlashForge Creator Pro, we choose the range for the geometric
parameters within the resolution and range of the machine, i.e.,
\(t_2 \in [0.4mm, 1mm]\) and \(L_x \in [50\mu m, \pm 300\mu m]\). The feasible range of \(K_1\) is chosen as
\(4 \cdot 10^{-4} \text{N/mm} \leq K_1 \leq 5 \cdot 10^{10} \text{N/mm}\).

The Active Set algorithm of Matlab is used for all optimization stud-
ies. Unlike most non-linear optimization tools, this algorithm estimates
a subset of constraints to watch while searching for the solution, rather
than imposing all constraints and bounds at every iteration. The gradi-
ents of the objective function and constraints are approximated using a
forward difference scheme. The function tolerance has been set to
\(10^{-6}\), the constraint tolerances to \(10^{-12}\), and the maximum number
of iterations to \(10^4\). These parameters offered excellent convergence for
our problem, while providing solutions in agreement with all
constraints.

This optimization process is repeated for three different values of
yield strain of the parent material (\(\epsilon_0 = [0.005, 0.01, 0.02]\)). For each
value of \(\epsilon_0\), an appropriate range for \(E/E_0\) is chosen, with the upper
bound limited by the no yielding constraint, and the lower bound
limited by manufacturability constraints. For \(\epsilon_0 = 0.005\), \(E/E_0 \in [3 \cdot 10^{-5}, 4 \cdot
10^{-7}]\); for \(\epsilon_0 = 0.01\), \(E/E_0 \in [3 \cdot 10^{-4}, 10^{-7}]\); and for \(\epsilon_0 = 0.02\), \(E/E_0 \in [3 \cdot
10^{-4}, 10^{-7}]\). To ensure that the algorithm converge to a global opti-
imum, each optimization study is repeated for 6 different values of the
initial guesses (5 randomly chosen values + the solution obtained for
the previous value of \(E/E_0\) and the independence of the solution on
the choice of initial guess is verified.)
3.2. Performance of optimal designs

The optimal results are shown in Fig. 8, plotting the optimal combinations of Young's modulus of the architected material (here normalized by the Young’s modulus of the parent material) and the loss factor, ψ. The scale is logarithmic on both axes. Recall that the product of Young’s modulus and loss factor represent our figure of merit, II. The three curves represent three different values of the yield strain of the parent material, εy. A number of important considerations can be made:

(i) The maximum value of the loss factor that can be achieved with this design is ψ = 0.5. The explanation is simple: we have defined ψ as the ratio of the area of the hysteresis loop to the area under the loading path, and we have the constrained the system to be reversible (i.e., all the hysteresis loop must be contained in the positive force quadrant). Clearly, the maximum area of a rhomboid (area of the hysteresis loop) that fits inside a triangle (area under the loading curve) is one half of the area of the triangle. It is important to realize that different publications utilize different definitions for the loss factor; as a result, alternative designs in other publications might have vastly different values of the optimal loss factors. The reader is encouraged to ensure consistent definitions, before attempting to compare optimal results.

(ii) All the optimal curves in Fig. 8 are clearly bi-linear, composed of a line of slope −1 and a vertical line at ψ = 0.5. All points in the −1 slope range have the same values of the figure of merit II = E · ψ. The explanation for this characteristic is illustrated in the insets in Fig. 8, where load-displacement curves are produced for three points on the optimal design curve: point 1 is in the −1 slope range, point 2 is at the intersection of the diagonal and vertical lines, and point 3 is in the vertical slope range. For all points on the −1 slope line, the no-yielding-of-the-K2 element constraint is active; hence, all the designs in that range have the same dimensions for the K1 and K2 springs, and thus the same value of II = E · ψ (see point 1 as an example). As the Young’s modulus, E, is decreased, K2 decreases while the area of the hysteresis loop is maintained constant, resulting in an increase in the loss factor (ratio of the dissipated energy to the applied energy).

When Fmin hits the x axis (point 2), we can no longer reduce K2 while maintaining the area of the hysteresis loop constant; this activates the constraint Fmin ≥ 0; from this point on, as we keep decreasing K3 (proportional to the Young’s modulus of the material), the area of the hysteresis loop is reduced while the loss factor remains constant at 0.5. Point 3 is representative of this regime. If the Young’s modulus of the architected material is reduced beyond point 2, the solution becomes sub-optimal, i.e. a solution can be found with the same value of the loss factor and a higher values of the Young’s modulus.

(iii) The yield strain of the parent material has a marked positive effect on the figure of merit.

(iv) With our choice of constituent materials and geometry, the maximum value of the figure of merit II = E · ψ is 5 · 10−5 · Es with Es the Young’s modulus of the parent material.

If we recall the design maps related for the generic 3-springs configuration (Fig. 4), we had observed that the figure of merit is independent on the value of K3/K1, as long as K3/K1 is sufficiently large. As K3/K1 is reduced while K2/K1 is maintained constant, the design eventually becomes irreversible. If one insists on reducing K3/K1 further beyond this critical value, K2/K1 must be reduced at the same time, in order to maintain recoverability. The relationship between those findings and the optimal design curve discussed in this section are clearly illustrated in Fig. 9: Point A is within a recoverable design range. As the Young’s modulus of the architected material is reduced between point A and point B, the system maintains a constant value of the figure of merit. At point B, the boundary delimiting the recoverability range is met; as the modulus is decreased further, from point B to point C, the optimal designs skirt the boundary of the recoverability range, and the solutions become sub-optimal.

One further remark can be made related to the observations from the 3-springs configuration analysis. As we have previously mentioned, a larger value of the K2 spring angle θ2 in a 3-spring configuration does not result in a larger value of the figure of merit. This is consistent with the optimal geometries for the architected material implementation: for all optimal designs, we find h/0.5l = 0.04 − 0.11 and h/t2 = 3 − 4; these values are not very high, and the ratio h/t2 is very close to the minimum

![Fig. 8](image-url) Optimized stiffness/damping response for the architected material implementation of the 3-spring configuration. Each curve represents a different value of yield strain of the constituent material. Note that the figure of merit, II, is constant along the diagonal section, until a value of ψ = 0.5 is reached. From that point forward, ψ remains constant and the stiffness drops, inducing a drop in the figure of merit. The insets qualitatively show force-displacement curves corresponding to three specific designs.
value required to have vertically symmetric buckling. In other words, steep arches are not required for optimal performance.

4. Conclusions

In this paper, we present a detailed mechanical analysis of a three-spring mechanical model under quasi-static cyclic loads, and show that the presence of an internal degree of freedom in the system provides a substantial amount of energy dissipation upon cyclic loading, even when fabricated with linear elastic materials with only limited intrinsic damping. The analytical model is used to rigorously investigate the effect of each geometric and materials parameter on the mechanical performance, and identify optimal geometries for maximum energy dissipation. In particular, we observe that proper selection of the ratio of the negative spring stiffness \( K_2 \) to the stiffness of the positive spring in series with it \( K_1 \) is crucial to ensure optimal energy dissipation: in particular, energy dissipation monotonically increases with increasing the ratio \( K_2/K_1 \), but an upper bound of \( K_2/K_1 \) is imposed by the recoverability constraint.

Compared to previously published snap-through-based designs for energy dissipation, which largely consisted of multiple identical bistable elements arranged in series [9,10,21,22], the proposed three-spring configuration presents multiple important advantages: (i) the addition of the spring \( K_2 \) allows for much stiffer designs; (ii) the addition of \( K_1 \) introduces an extra degree of freedom, which can be used to maximize the amount of energy dissipation or carefully tune the snap-through displacement; (iii) the combined addition of \( K_1 \) and \( K_2 \) offers the possibility to design lossy architected materials with recoverable behavior, even when \( K_2 \) is bistable (as discussed in this work); (iv) the combined presence of \( K_2 \) and \( K_3 \) ensures that hysteresis occurs even within a single unit cell, whereas traditional \( K_2 \) designs require a large number of unit cells for energy dissipation [9].

Subsequently, we present a particular implementation of the 3-spring model as the unit cell of a periodic architected material. The specific configuration is chosen so that it can be easily fabricated in a single base material by additive manufacturing techniques, e.g., brazing or 3D printing, and do not require any external pre-compression of the springs, or any pin joints, in order to achieve negative stiffness. All springs are implemented by arches, resulting in an architected material that is bending dominated under the applied loads considered in this study. This allows for substantial tunability of the range of motion, a critical condition for several structural applications. Conceptually similar stretching dominated configurations are currently under development and will be presented elsewhere.

The analytical model developed for the generic 3-spring mechanical model is applied to the particular implementation, and the predicted load-displacement curve obtained under cyclic loading is verified with numerical models (Finite Elements Analyses) and validated with a cyclic compression experiment on a 3D printed prototype. The validated analytical model is then used for optimal design studies, where architected materials with optimal values of the figure of merit \( H \), defined as the product of Young’s modulus, \( E \) and loss coefficient, \( \psi \). We show that the maximum achievable value of the figure of merit is proportional to the Young’s modulus and to the yield strain of the constituent material. Another important remark is that for our specific architected material implementation, the limiting constraint is the yield strength of the \( K_2 \) arch; clearly, the thicker the arch, the more difficult it will be to achieve snap-through without yielding. Once the optimal dimensions of the \( K_2 \) arch are obtained, the figure of merit could be further increased by stacking multiple arches of identical dimensions on top of each other; as each arch would experience the same maximum stress (which in an optimized system would be equal to the yield strength of the material), the constraint would not be violated, whereas the figure of merit would increase proportionally to the number of arches. Clearly this assumes that the stiffness of the spring \( K_2 \) could be increased proportionally to \( K_2 \) as needed without violating manufacturing constraints, and that the recoverability constraint can still be met (i.e., \( \psi < 0.5 \)). An alternative approach for further increasing the figure of merit is replacing the arch with a structural element with higher negative stiffness.

In conclusion, we demonstrate a flexible approach to architected material design that allows for tailorible combinations of high stiffness, low density and high energy dissipation by enhancing the host materials inherent damping. The proposed architected material can be designed to exhibit single or bi-stability and can be implemented with any constituent material at virtually any length scale. Several additive manufacturing approaches can be adopted for fabrication, depending on the constituent material and the unit cell size. The energy dissipation mechanism employed here does not require additional damping specific materials, nor does it require a large number if unit cells, and the proposed architected material can be fabricated from virtually any constituent material, potentially allowing use under high temperatures or aggressive environmental conditions. Future developments include a stretching dominated lattice material implementation, for improved performance.

Acknowledgments

Funding from Office of Naval Research (Program Manager: D. Shiffer, Contract No. N000141110884) is gratefully acknowledged. AGI would like to thank the Obra Social “la Caixa” for partial financial support. AGI and RFA are also thankful to the Balsells Foundation for partial financial support. The Abaqus Finite Element Analysis software is licensed from Dassault Systemes SIMULIA, as part of a Strategic Academic Customer Program between UC Irvine and SIMULIA.
Appendix A. Stiffness and strength of the arch-shaped springs

Here we present the approach that is used to obtain close-form solutions for the stiffness of elements $K_1$ and $K_3$ (the positive stiffness elements) in the cellular material implementation of the 3-spring model. As both springs are arches, which are primarily bending dominated, we invoke dimensional analysis supported by beam theory to conclude that the stiffness of each arch will be proportional to the Young’s modulus of the constituent material, $E_s$, the width of the arch and the cube of the thickness/length ratio, i.e.:

$$
\frac{K_1}{E_s \cdot W_1} = a \left(\frac{t_1}{l_1}\right)^3
$$

(A.1)

$$
\frac{K_3}{E_s \cdot W_3} = b \left(\frac{t_3}{R_3}\right)^3
$$

(A.2)

where $a$ and $b$ are non-dimensional coefficients. The numerical values of the two coefficients $a$ and $b$ are obtained by fitting of Finite Elements analyses performed in ABAQUS. Both arches are modeled with 2D solid elements with linear elastic material, and subjected to an end displacement in the vertical direction equal to 0.2, and subjected to an end displacement in the vertical direction equal to 0.2 $l_1$ and 0.4 $R_3$ for $K_1$ and $K_3$, respectively. All analyses are quasi-static and non-linear. A number of simulations are performed with arches of different thickness, and the axial stiffness of the arch (force/displacement at the end of the loading cycle) is plotted against the arch thickness. A cubic fit provided the values of the parameters $a$ and $b$, namely $a = 1.38$ and $b = 0.46$ (See Fig. A1). Notice that the cubic fit expressed in Eqs. (A.1)–(A.2) is in excellent agreement with the FE results.

The strength of both elements was also modeled by dimensional analysis using beam theory, with fitting coefficients obtained by Finite Elements simulations. The geometry of element $K_1$ can be simplified as a quarter of a circle, with the bottom end clamped and the top end allowed to freely rotate, as schematically depicted in Fig. A2. The maximum stress will clearly occur at the mid-section, on the inside of the arch. If we assume that the whole element is under an axial displacement $Z$, the applied force at the mid-section (quarter of the circle) will be $F = K_1 \cdot Z$. The maximum stress is a combination of the axial and bending stress, and can be expressed as:

$$
\sigma_{\text{max}}^{K_1} = \frac{F}{A} + \frac{F \cdot R_1}{l_1} \frac{l_1}{2} \frac{K_1 \cdot Z}{W_1 \cdot l_1} \frac{K_1 \cdot Z \cdot 0.25 l_1}{I} Z
$$

$$
= 2.06 \frac{F l_1}{l_1} + 1.38 \frac{F l_1}{l_1} Z
$$

(A.3)

The element $K_3$ can be simplified as a quarter of circle, with the bottom clamped and the top end with the rotation constrained; this results in the establishment of a moment $M_0$, as plotted in Fig. A2. The maximum stress will occur at the top end, and can be expressed as:

$$
\sigma_{\text{max}} = \frac{M_0}{Z}
$$

(A.4)

We find the moment $M_0$ using Castigliano’s theorem, imposing $\delta_0 = 0$ at the top end. Consequently we find:

$$
\sigma_{\text{max}}^{K_3} = \frac{3}{\pi} \frac{F l_3}{R_3^3} Z
$$

(A.5)

The stiffness and maximum stress experienced by the arch $K_3$ (the negative stiffness element) were reported in [28].

Appendix B. Mechanical characterization of the base material (ABS) printed by Fused Deposition Modeling

A prototype of the unit cell of the proposed periodic cellular material implementation is printed by Fused Deposition modeling in ABS using a Flashforge Creator Pro (see Section 2.4 for details). In order to correctly characterize our prototype and ensure that none of the elements yield under the design loading-unloading cycle, it is essential to measure the base material properties. Dog-bone tensile specimen in agreement with the ASTM D638–014 standard are printed with the same Flashforge Creator Pro used for all archited material fabrication. All arch elements are printed separately to ensure fiber orientation along the length of the arch; hence, dogbone test coupons are only printed with fibers aligned with the loading direction. The Young’s modulus of printed ABS is extracted using the strain measured by a strain gage (Fig. B1), and is reported as $E = 1.74$ GPa. The yield strain of printed ABS is measured using the displacement reading of the universal testing
machine and the 0.2% residual strain rule (Fig. B1), and is reported as \( \varepsilon_y = 0.016 \). To ensure that our material remains elastic throughout the complete loading-unloading cycle, we will impose that no element ever reach a strain higher than \( \varepsilon_y = 0.01 \).

As explained in Section 3.1, the optimization scheme proceeds as follows: the effective Young’s modulus of the lattice, \( E \), is swept in the chosen range. For every value of \( E \), the optimizer identifies the best combination of 5 independent variables (\( K_1 (N/mm) \), \( K_3 (N/mm) \), \( W_2 (mm) \), \( t_2 (mm) \) and \( h (mm) \)) that maximizes the loss coefficient, \( \psi \). The Young’s modulus of the parent material is set at \( E_s = 2 \) GPa, and the optimization studies are repeated for three different values of the yield strain of the parent material (\( \varepsilon_y = 0.005 \), 0.01, and 0.02). The length of the bi-stable element, \( L \) is kept constant at 100 mm.

The optimization results are reported in Table A1 (some intermediate values were omitted for conciseness). Notice that a decrease in \( E \) induces a simultaneous decrease in \( K_1 \), \( K_3 \), \( h \) and \( t_2 \). Also notice that for all optimal designs, \( h/0.5L = 0.04 - 0.11 \) and \( h/t_2 = 3 - 4 \).

Once the optimal values of the primary variables are determined, Eqs. (A.1) and (A.3) are used to select suitable values of \( t_1 \), \( W_1 \), \( l_1 \) and Eqs. (A.2) and (A.5) are used to select suitable values of \( t_3 \), \( W_3 \), \( R_3 \).

Table C1
Optimal geometries leading to the results summarized in Fig. 8. Please refer to Fig. 5a for variables definition (several intermediate rows were omitted for conciseness).

<table>
<thead>
<tr>
<th>( E_s \times 10^{-5} )</th>
<th>( \psi )</th>
<th>( K_1 (N/mm) )</th>
<th>( K_3 (N/mm) )</th>
<th>( W_2 (mm) )</th>
<th>( t_2 (mm) )</th>
<th>( h (mm) )</th>
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